CSE Qualifying Exam, Spring 2019: Numerical Analysis

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- Please answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for clarity as well as correctness.

Questions:

1. Suppose a matrix $H \in \mathbf{R}^{n\times n}$ is of the form $H = I - \alpha hh^T$ where $\|h\|_2 = 1$.
   (a) Find a value (or values) of α that makes H orthogonal.
   (b) One claims that for any given vector $y \in \mathbf{R}^{n\times 1}$, there is an orthogonal matrix H of the above form with which $Hy = (\,0 \quad 0 \quad \cdots \quad 0 \quad \beta\,)^T$ where β is a scalar. If this claim is true, then show how you would compute the vector h. If your answer is no, then present a counter example.
   (c) Describe how you would solve a linear system $Ax = b$, where $A \in \mathbf{R}^{n\times n}$ is nonsingular, using the transformations of the above form.

2. (a) Does a singular value decomposition (SVD) exist for any matrix $A \in \mathbf{R}^{m\times n}$? If your answer is yes, prove it. If your answer is no, then explain why.
   (b) Assume that $m = n$, $A^T = A$ and consider a SVD of A and an eigenvalue decomposition (EVD) of A. Under what conditions would SVD be the same as EVD? Justify your answer.

3. Consider a linear system $A\vec{x} = \vec{b}$ with A being a $m \times m$ symmetric positive definite matrix. Given an approximation $\vec{x}_0$ and a subspace K of $\mathbb{R}^m$, denote $\vec{x}_1$ as the minimizer of the following optimization problem,

$$\vec{x}_1 = argmin_{\vec{x}\in\{\vec{x}_0+K\}}(\vec{x}^* - \vec{x})^T A(\vec{x}^* - \vec{x}),$$

   where $\vec{x}^*$ is the exact solution of $A\vec{x} = \vec{b}$.
   (a) Prove that $\vec{r}_1 = \vec{b} - A\vec{x}_1$ is orthogonal to K.
   (b) If one selects $K = span\{\vec{r_0}\}$, where $\vec{r_0} = \vec{b} - A\vec{x}_0$, derive a formula for $\vec{x}_1$.
   (c) Design an iterative algorithm, using the formula you obtained in (b), to compute solution of $A\vec{x} = \vec{b}$. What is the computational cost in each iteration of your algorithm?

4. Suppose A is a 2010001×2010001 symmetric positive definite matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$.
   (a) Give the sharpest lower bound on the condition number of A.
   (b) One uses Gaussian Elimination with partial pivoting to solve the linear system $A\vec{x} = \vec{b}$. What is your estimate for the relative error of the numerical solution, assuming the machine error is 10^{-16}. You must justify your answer.