CSE Qualifying Exam, Fall 2020: Numerical Methods

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, and/or internet usage allowed at any time during the exam.
- Please answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for clarity as well as correctness.
- Good luck!

Question 1:

(a) Let

$$A = \begin{bmatrix} 1 & 1-4\epsilon \\ 1+4\epsilon & 1 \end{bmatrix},$$

where $0 < \epsilon < \frac{1}{4}$. What is the condition number of matrix A, measured in ∞−norm?

(b) Give the definition of backward stability for an algorithm.

(c) Given a matrix B with condition number 10^4, what is the expected relative error in the solution if one tries to solve $B^T B\vec{x} = B^T\vec{b}$ by a backward stable algorithm? (Assume that the machine precision is 10^{-16}.)

Question 2: Consider a matrix

$$A = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Show that A is positive definite.

(b) Calculate the $A-$norm of the vector

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(c) Find a vector that is conjugate ($A-$orthogonal) to $\vec{b}$.

(d) Let $B \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $\vec{c} \in \mathbb{R}^n$ a given vector. Find the linear system whose solution is the minimizer of

$$\min_{\vec{x}}\{\vec{x}^T B\vec{x} - \vec{c}^T\vec{x}\}.$$

You must show the steps to obtain you result.

Question 3: Let $P \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

Question 4: Let $A \in \mathbb{C}^{m\times m}$ be a nonsingular matrix. For solving the system of equations $A\vec{x} = \vec{b}$, consider the method

$$\vec{x}_{k+1} = \vec{x}_k + \alpha\vec{r}_k, \quad \vec{r}_k = \vec{b} - A\vec{x}_k$$

where α is a scalar constant, starting with an initial approximation $\vec{x}_0$.

(a) Let $p_k(A)$ be the matrix polynomial at step k such that $\vec{r}_k = p_k(A)\vec{r}_0$. What is this polynomial?

(b) Suppose that the eignvalues of A lie in a circle in the complex plane centered at $z = 2$ with radius $1/2$. What choice of α do you recommend for the method? What would you expect to be the corresponding convergence rate?