There are four problems below. Please choose three to solve. If you choose to solve all four, only the lowest three scores will count. Show all your work and write in a readable way.

Question 1

Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 6 & -3 & 2 \\ 3 & -2 & 1 \\ 2 & -1 & 1.2 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 2 \\ 2.0 \end{pmatrix}$$

The condition number of the matrix A is $\kappa_1(A) = 33$.

(a) Consider the vector $\tilde{x} = (1, 1, 1)^T$ as an approximate solution to the system. Using the information given, obtain an upper bound for $\|x - \tilde{x}\|_1 / \|x\|_1$, where x is the exact solution of the system. In general, we do not know the exact solution, and this question is to be answered under the assumption that we do not know the exact solution. You will not get any credit if you actually obtain the exact solution.

(b) Show that if the term $a_{33} = 1.2$ is replaced by $a_{33} = 0.6666... = 2/3$ the matrix A becomes singular. You will not any credit if you answer this question by computing a determinant.

(c) Use the result of the previous question to find a lower bound for $\kappa_1(A)$. Compare with the condition number given above and verify that you indeed obtained a lower bound.

Question 2

For a Hermitian matrix, the Lanczos algorithm is often used to compute the extremal eigenvalues. Formulate a theorem and prove in what sense the Lanczos algorithm can or cannot be used to compute an accurate lower bound and upper bound on the largest eigenvalue.

Question 3

Consider a symmetric, nonsingular matrix

$$H = \begin{bmatrix} 0 & B \\ B^T & A \end{bmatrix}$$

where B has dimensions $m \times n$ with $m \geq n$. Give a backward stable *finite* algorithm for computing the factorization

$$H = QMQ^T$$

where Q is orthogonal and M has the form

$$M = \begin{bmatrix} 0 & 0 & Y^T \\ 0 & X & Z^T \\ Y & Z & W \end{bmatrix}$$

where X is symmetric positive definite, W is symmetric, and Y is square and lower anti-triangular.

The anti-diagonal of a matrix is the diagonal from the top-right to the bottom-left. A lower anti-triangular matrix is zero above the anti-diagonal, i.e., the $r \times r$ matrix Y is lower anti-triangular if $Y_{ij} = 0$ for $i + j \leq r$.

Your algorithm must be *finite*, meaning that it completes after a fixed number of steps. In other words, your algorithm cannot compute eigenvalues or singular values.

Question 4

Consider the two-point boundary value problem

$$\frac{d^2u}{dx^2} + u = -x, \quad 0 \le x \le 1$$

with boundary conditions $u(0) = 0$ and $u(1) = 0$. Consider solutions of the form

$$v(x) = x(1-x)(a_1 + a_2 x)$$

where a_1 and a_2 are parameters to be determined.

(a) What is a general advantage of the Finite Element Method over the Finite Difference Method for solving two-point boundary value problems (or partial differential equations in general)?

(b) Use the method of weighted residuals to solve the boundary value problem. Use the weight functions $w_1(x) = x(1-x)$ and $w_2(x) = x^2(1-x)$.

(c) Use collocation to solve the boundary value problem. Use the two collocation points $t_1 = 0.5$ and $t_2 = 0.75$. Do not use the boundary conditions when constructing equations to solve. Why are the boundary conditions not necessary in your solution method?