CSE Qualifying Exam, Fall 2020: Numerical Analysis

Instructions:

- This is a CLOSED BOOK exam. No books or notes are allowed.
- No calculators, computers, phones, or internet usage allowed at any time during the exam (except for purposes of electronic proctoring, e.g., Honorlock).
- Answer three of the following four questions. All questions are graded on a scale of 10. If you answer all four, all answers will be graded and the three lowest scores will be used in computing your total.
- Show all your work and write in a readable way. Points will be awarded for correctness as well as clarity.
- Good luck!

Question 1: Consider a linear system of equations

$$Ax = b,$$

where

$$A = \begin{bmatrix} 2 & 0 & \alpha \\ 0 & 2 & 0 \\ \alpha & 0 & 2 \end{bmatrix}.$$

(a) Find all values of α such that A is symmetric positive definite.

(b) What is the iteration matrix if Gauss-Seidel iteration is applied to solve this linear system?

(c) Find all values of α such as the Gauss-Seidel iteration is convergent.

You must justify your answers.

Question 2: Consider $A \in \mathbb{R}^{n \times n}$.

- Give the algorithm to use inverse iteration with shift μ to compute an eigenvalue of A.
- Apply the algorithm to the following matrix A with $\mu = 4.5$,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

What is the eigenvalue that the algorithm computes? Justify your answer.

Question 3:

(a) Prove or disprove: the product of two symmetric matrices is symmetric.

(b) Given two matrices, A and B, that are symmetric and positive definite, and given that

$$c_1 u^T B u \leq u^T A u \leq c_2 u^T B u$$

for positive constants c_1 and c_2 and all vectors $u \neq 0$, prove that the condition number of $B^{-1}A$ is not greater than c_2/c_1.

Hint: You may use the fact that the smallest and largest eigenvalues of a symmetric matrix M satisfies

$$\lambda_{\min} = \min_{x \neq 0} \frac{x^T M x}{x^T x} \qquad \text{and} \qquad \lambda_{\max} = \max_{x \neq 0} \frac{x^T M x}{x^T x},$$

respectively.

Question 4: Given a symmetric positive definite matrix with a 2×2 block partitioning,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

show how to construct a triangular matrix R such that

$$R^T A R = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}.$$